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GREEN'S TENSOR AND THE BOUNDARY INTEGRAL EQUATIONS FOR THIN ELASTIC MULTILAYER ASYMMETRIC ANISOTROPIC PLATES†

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The matrix of quasistatic fundamental solutions of the averaged equations of elasticity for a thin multilayer plate of arbitrary asymmetric construction with general anisotropy of the layers is constructed. The main difference from the classical case arises when analysing the closely associated processes of bending and tension–compression–shear since, generally speaking, these plates do not contain a neutral plane (a plane that remains undeformed during bending). The reciprocity theorem in steady-state dynamics and statics is used to obtain integral identities. For the main types of related boundary-value problems of statics, a system of four boundary integral equations is derived. The singularities of the kernels are studied and the properties of the equations are investigated.

1. The main purpose of this paper is to construct boundary integral equations (BIE) for the basic boundary-value problems of associated bending and tension–compression–shear of a thin multilayer elastic plate. The association is due to the asymmetrical arrangement of the layers over the depth and, at the same time, the substantial anisotropy of the layer material. In physical terms this means that, generally speaking, every longitudinal plane bends and deforms and there is no neutral plane. We consider the most general case where both the equations and the boundary conditions are closely associated, so that the additional terms are of the same order as the classical terms.

A detailed analysis and classification of the equations and boundary-value problems has been given in [1–4]. In [3, 5] the problems of statics were solved by introducing complex potentials for the displacements, which enabled an analytic technique to be used akin to the method of Kolosov, Myskhelishvili and Lekhnitskii [6–8]. However, in some cases it is preferable to use BIE. A thorough study has been made of their use in individual bending and planar problems [7–11], but not in associated situations.

There are quite a few algorithms for the numerical solution of various BIE (the method of boundary elements, the method of expansion in terms of special polynomials, etc.) [9–12]. It is therefore possible to use standard procedures. The method of complex potentials previously employed [5] is less standardized in this sense, although it is easy to use to obtain solutions in closed form for a number of canonical regions.

2. We consider the stress–strain state (SSS) inside (far from the edges of) a thin packet of N ideally coupled elastic layers. The layers are arbitrarily arranged over the depth. For instance, they might be asymmetrically arranged about the mid-plane of the packet. It is assumed that each jth layer $j = 1, 2, \ldots, N$ has a constant thickness h_j and density ρ_j, and its elastic properties are described by the three-dimensional Hooke's law with stiffness ratio $\mathbf{G}_j$. The general anisotropic case where each matrix $\mathbf{G}_j$ contains 21 independent constants is allowed.

We shall assume that the half-thickness of the packet h is much smaller than the characteristic scale of deformation L in a longitudinal direction $\varepsilon = h/L \ll 1$. We shall also assume that the ratios of the densities, the elastic and geometric parameters of the layers are incommensurate with ε, that is, are of order one as $\varepsilon \to +0$, and the characteristic time of the dynamic process is quite large and of order not less than $O(\varepsilon^{-1})$. We denote the transverse and longitudinal Cartesian coordinates by $x_3 = z$, $\mathbf{x} = \mathbf{i}_\alpha x_\alpha$ $(\alpha = 1, 2)$, respectively, $z_j < z_{j+1}$ are the coordinates of the front surfaces of the jth layer, and Ω is the region that the plate occupies in plan. Suppose that the normal and shear stresses on the front surfaces of the plate are given in the form

$$\sigma_{33}^{\mp} = \sigma^{\mp}(\mathbf{x}, t), \quad \sigma_{\alpha 3}^{\mp} = \tau_{\alpha}^{\mp}(\mathbf{x}, t) \quad (z^{-} = z_1, \; z^{+} = z_{N+1}) \tag{2.1}$$

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where $\sigma^{\mp} = O(1)$, $\tau^{\mp} = O(\varepsilon^{-1})$. The last condition is imposed for convenience, since it ensures that the limiting orders of response of the plate to shear and normal loads are equal.

If the layers are both asymmetrically stacked and anisotropic, the asymptotic principal part of the SSS (with relative error $O(\varepsilon)$) satisfies the following elasticity relations

$$W_j = w(\mathbf{x},t), \quad \mathbf{U}_j = \mathbf{u}(\mathbf{x},t) + z\boldsymbol{\theta}$$

$$(\boldsymbol{\theta} \equiv \mathbf{i}_\alpha \theta_\alpha, \ \theta_\alpha \equiv -\partial_\alpha w, \ \partial_\alpha \equiv \partial / \partial x_\alpha)$$

$$e_{\alpha\beta} = \varepsilon_{\alpha\beta} + z\theta_{\alpha\beta}, \quad \varepsilon_{\alpha\beta} = \tfrac{1}{2}(\partial_\beta u_\alpha + \partial_\alpha u_\beta), \quad \theta_{\alpha\beta} = \partial_\alpha \theta_\beta$$

$$\sigma^j_{\alpha\beta} = \chi_{\alpha\beta}(\boldsymbol{\Gamma}_j)(\mathbf{u} + z\boldsymbol{\theta}), \quad (Q_{\alpha\beta}, M_{\alpha\beta}) = \sum_j \int_{z_j}^{z_{j+1}} (1,z)\sigma^j_{\alpha\beta} dz$$

$$Q_{\alpha 3} = \partial_\beta M_{\alpha\beta}, \quad (\alpha,\beta = 1,2; \ 1 \leftrightarrow 2)$$

$$\boldsymbol{\chi}_{11} = \mathbf{i}_1(\gamma_{11}\partial_1 + \gamma_{16}\partial_2) + \mathbf{i}_2(\gamma_{16}\partial_1 + \gamma_{12}\partial_2)$$

$$\boldsymbol{\chi}_{12} = \mathbf{i}_1(\gamma_{16}\partial_1 + \gamma_{66}\partial_2) + \mathbf{i}_2(\gamma_{66}\partial_1 + \gamma_{26}\partial_2)$$

$$\boldsymbol{\Gamma} = \begin{Vmatrix} \gamma_{11} & \gamma_{16} & \gamma_{12} \\ \gamma_{16} & \gamma_{66} & \gamma_{62} \\ \gamma_{21} & \gamma_{26} & \gamma_{22} \end{Vmatrix}, \quad \gamma_{pq} = \frac{\det(\mathbf{G}^p_q)}{\det(\mathbf{G}_0)}, \quad \mathbf{G}_0 = \mathbf{G}\binom{345}{345}$$

Here W_j and $\mathbf{U}_j$ are the deflection and longitudinal displacements (and are independent of the layer number), $\sigma^j_{\alpha\beta}$ are the stresses, $\varepsilon_{\alpha\beta}$ and $\theta_{\alpha\beta}$ are the deformations and curvatures in the plane $z = 0$, $Q_{\alpha\beta}$ and $M_{\alpha\beta}$ are the longitudinal forces and moments in the plate section, and $Q_{\alpha 3}$ is the shearing force. For convenience in the calculations, we have slightly changed the expression for the shearing force of [5] and omitted the terms $z^+\tau^+_\alpha - z^-\tau^-_\alpha$ on the right.

The indices 4, 5, 6 in the initial 6 × 6 stiffness matrices **G** correspond to stresses with indices 23, 13, 12. The matrices **G** are formed from the average stiffnesses of the current layer, $\mathbf{G}_0$ is the principal minor in the initial matrix **G**, and the minor $\mathbf{G}^p_q$ is obtained by bordering the minor $\mathbf{G}_0$ by the pth row and qth column below and to the right. These relations are discussed in more detail in [2–4]. The basic equations take the form [2, 3, 5]

$$\begin{aligned} &\partial_\beta \boldsymbol{\chi}_{\alpha\beta}(\mathbf{D}_1)\mathbf{u} - \partial_\beta \boldsymbol{\chi}_{\alpha\beta}(\mathbf{D}_2)\,\mathrm{grad}\, w + T_\alpha = 0 \\ &-\partial^2_{\alpha\beta}\boldsymbol{\chi}_{\alpha\beta}(\mathbf{D}_2)\mathbf{u} + \{\partial^2_{\alpha\beta}\boldsymbol{\chi}_{\alpha\beta}(\mathbf{D}_3)\,\mathrm{grad} + \rho_*\partial^2_t\}w = T_3 \end{aligned} \tag{2.2}$$

$$\rho_* = \sum_j \int_{z_j}^{z_{j+1}} \rho_j dz, \quad \mathbf{D}_k = \sum_j \int_{z_j}^{z_{j+1}} z^{k-1}\boldsymbol{\Gamma}_j dz \tag{2.3}$$

$$T_\alpha = \tau^+_\alpha - \tau^-_\alpha, \quad T_3 = \sigma^+ - \sigma^- + \mathrm{div}(z^+\boldsymbol{\tau}^+ - z^-\boldsymbol{\tau}^-)$$

The difference from the classical relations of the theory of plates of symmetric construction is that the equations of bending and the plane stressed state (2.2) are associated. In both the equations and the forces (moments), the bending ($\mathbf{D}_3$) and membrane stiffnesses ($\mathbf{D}_1$) are accompanied by non-zero mixed stiffnesses ($\mathbf{D}_2$) ($\delta^\beta_{\alpha+1}$ is the Kronecker delta, $\alpha\beta = 11, 12, 22$)

$$\begin{Vmatrix} Q_{\alpha\beta} \\ M_{\alpha\beta} \end{Vmatrix} = \begin{Vmatrix} \mathbf{D}_1 & \mathbf{D}_2 \\ \mathbf{D}_2 & \mathbf{D}_3 \end{Vmatrix} \begin{Vmatrix} \dot{\nu}_{\alpha\beta} \\ \varkappa_{\alpha\beta} \end{Vmatrix}, \quad \begin{Vmatrix} \nu_{\alpha\beta} \\ \varkappa_{\alpha\beta} \end{Vmatrix} \equiv \begin{Vmatrix} \varepsilon_{\alpha\beta} \\ \theta_{\alpha\beta} \end{Vmatrix} (1 + \delta^\beta_{\alpha+1}) \tag{2.4}$$

Correspondingly the boundary conditions on $\partial\Omega$ [4, 5] are assigned jointly for the two types of equation ($\mathbf{n}$ and $\boldsymbol{\tau}$ are the unit vectors of the outward normal and the tangent, with $\partial\Omega$ traversed in a positive direction):

the first boundary-value problem—u_n, u_τ, $\theta_n = -\partial_n w$, w are given;
the second boundary-value problem—the longitudinal forces Q_n and Q_τ, the bending moment M_n and Kirchhoff's shearing force $P_n = Q_n + \partial_\tau M_\tau$ are given.

One might also consider "crossed" (of the form u_n, Q_τ, θ_n, P_n, for example) or mixed boundary conditions. In the Cauchy problem, the initial conditions at $t = 0$ are set only for the functions w and $\partial_t w$.

3. Suppose that the plate oscillates harmonically as $e^{i\omega t}$ (this time factor is omitted below). Let $\mathbf{V} = (u_\alpha, w)$ and $\mathbf{V}^\mu = (u^\mu_\alpha, w^\mu)$ denote any two sets of displacements which give an SSS of a finite or infinite plate and correspond to individual loads $\mathbf{T} = (T_\alpha, T_3)$ and $\mathbf{T}^\mu = (T^\mu_\alpha, T_3^\mu)$; the boundary conditions will be specified later. We will write the reciprocity relations which both SSS must satisfy. These will be of importance later, and so we shall discuss them in detail. Consider the "crossed" potential energy density π, the kinetic energy τ, the work of individual loads a and the vector $\boldsymbol{\eta} = \mathbf{i}_\alpha \eta_\alpha$ (α, $\beta = 1, 2$)

$$\pi(\mathbf{V}, \mathbf{V}^\mu) = \tfrac{1}{2}\left(\varepsilon_{\alpha\beta} Q^\mu_{\alpha\beta} + \theta_{\alpha\beta} M^\mu_{\alpha\beta}\right), \quad \tau(\mathbf{V}, \mathbf{V}^\mu) = -\tfrac{1}{2}\rho_* \omega^2 w w^\mu$$

$$a(\mathbf{V}, \mathbf{T}^\mu) = u_\alpha T^\mu_\alpha + w T^\mu_3, \quad \eta_\alpha(\mathbf{V}, \mathbf{V}^\mu) = u_\beta Q^\mu_{\alpha\beta} + \theta_\beta M^\mu_{\alpha\beta} + w Q^\mu_{\alpha 3}$$

The reciprocity theorem. For any two states $\mathbf{V}$ and $\mathbf{V}^\mu$ of an asymmetrically multi-layered anisotropic plate, the following relations are satisfied

$$\begin{gathered} \pi(\mathbf{V}, \mathbf{V}^\mu) + \tau(\mathbf{V}, \mathbf{V}^\mu) = \pi(\mathbf{V}^\mu, \mathbf{V}) + \tau(\mathbf{V}^\mu, \mathbf{V}) \\ a(\mathbf{V}, \mathbf{T}^\mu) - a(\mathbf{V}^\mu, \mathbf{T}) = \operatorname{div} \boldsymbol{\eta}(\mathbf{V}, \mathbf{V}^\mu) - \operatorname{div} \boldsymbol{\eta}(\mathbf{V}^\mu, \mathbf{V}) \end{gathered} \tag{3.1}$$

$$\begin{gathered} \Pi(\mathbf{V}, \mathbf{V}^\mu) + \mathcal{T}(\mathbf{V}, \mathbf{V}^\mu) = \Pi(\mathbf{V}^\mu, \mathbf{V}) + \mathcal{T}(\mathbf{V}^\mu, \mathbf{V}) \\ A(\mathbf{V}, \mathbf{T}^\mu) - A(\mathbf{V}^\mu, \mathbf{T}) = \int_{\partial\Omega} \{u^\mu_n Q_n - u_n Q^\mu_n + u^\mu_\tau Q_\tau - \\ -u_\tau Q^\mu_\tau + \theta^\mu_\eta M_\eta - \theta_n M^\mu_n + w^\mu P_n - w P^\mu_n\} dl \end{gathered} \tag{3.2}$$

$$(\Pi, \mathcal{T}, A) = \int_\Omega (\pi, \tau, a) d\Omega$$

The proof can be obtained from the general theorem for a three-dimensional elastic body (applied to the asymptotic principal part of the SSS) or by transforming Eqs (2.2), written in terms of the force and momentum

$$\partial_\beta Q_{\alpha\beta} + T_\alpha = 0$$

$$\partial^2_{\alpha\beta} M_{\alpha\beta} - \rho_* \omega^2 w \equiv \partial_\alpha Q_{\alpha 3} - \rho_* \omega^2 w = T_3$$

to the form

$$2\{\pi(\mathbf{V}, \mathbf{V}^\mu) + \tau(\mathbf{V}, \mathbf{V}^\mu)\} = a(\mathbf{V}, \mathbf{T}^\mu) - \operatorname{div} \boldsymbol{\eta}(\mathbf{V}, \mathbf{V}^\mu) \tag{3.3}$$

By virtue of Eqs (2.4) and the symmetry of the stiffness matrices, the expressions for the energy densities (3.3) are symmetric bilinear forms for the deformations (curvatures) or forces (moments), whence Eqs (3.1) and (3.2) follow.

Note that when multiplying by the frequency $i\omega$, we formulate Eqs (3.1) and (3.2) for the powers in an obvious way (without the factor 2 on the left-hand sides), while the vector $\boldsymbol{\eta}$ will give the power flux density.

4. We will derive the BIE by the standard method [8, 9]. Let $\mathbf{V}^\mu = \mathbf{V}^\mu(\mathbf{x}, \mathbf{x}_0)$ be a formal solution of Eqs (2.2) for an infinite plate with individual loads in the form of delta functions. Then from relations (2.1), (2.3) and (3.2) for

$$\tau_\alpha^\mp = \mp \tfrac{1}{2}\delta_\alpha^\mu \delta(\mathbf{x}'),\quad \sigma^\mp = \mp \tfrac{1}{2}\delta_3^\mu \delta(\mathbf{x}')$$

$$T_\alpha^\mu = \delta_\alpha^\mu \delta(\mathbf{x}'),\quad T_3^\mu = \delta_3^\mu \delta(\mathbf{x}') + z_0 \delta_\alpha^\mu \partial_\alpha \delta(\mathbf{x}') \tag{4.1}$$

$$\mathbf{x}' = \mathbf{x} - \mathbf{x}_0,\quad \mathbf{x}_0 \in \Omega,\quad z_0 = \tfrac{1}{2}\left(z^+ + z^-\right)$$

we obtain the relations

$$\delta_3^\mu w(\mathbf{x}_0) + \delta_\alpha^\mu \{u_\alpha(\mathbf{x}_0) + z_0 \theta_\alpha(\mathbf{x}_0)\} = \int_\Omega \{u_\alpha^\mu T_\alpha + w^\mu T_3\}\Omega +$$

$$+ \int_{\partial\Omega} \{u_n^\mu Q_n - u_n Q_n^\mu + u_\tau^\mu Q_\tau - u_\tau Q_\tau^\mu + \theta_n^\mu M_n -$$

$$-\theta_n M_n^\mu + w^\mu P_n - w P_n^\mu\} dl \tag{4.2}$$

From formula (4.2) we can determine the displacements at any point $\mathbf{x}_0$ if the displacements and loads on the boundary $\partial\Omega$ are known. Letting the point $\mathbf{x}_0$ approach the boundary $\partial\Omega$, we obtain the BIE for the unknown boundary conditions. Using the above scheme, we obtain the fundamental solutions and BIE for problems of statics.

5. We will find the Fourier transforms of the fundamental solutions of the equations of statics (2.2) with the right-hand sides (4.1) (omitting the superscript μ in obvious cases)

$$f^*(\mathbf{s}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mathbf{x}') e^{i\mathbf{s}\mathbf{x}'} dx_1' dx_2',\quad \mathbf{s} = \mathbf{i}_\alpha s_\alpha$$

$$f(\mathbf{s}) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} e^{-is_1 x_1'} ds_1 \int_{-\infty}^{\infty} f^*(\mathbf{x}') e^{-is_2 x_2'} ds_2 \tag{5.1}$$

We obtain

$$\begin{Vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{Vmatrix} \begin{Vmatrix} u_1^* \\ u_2^* \\ iw^* \end{Vmatrix} = \begin{Vmatrix} T_1^* \\ T_2^* \\ iT_3^* \end{Vmatrix},\quad \begin{matrix} T_\alpha^* = \delta_\alpha^\mu \\ T_3^* = \delta_3^\mu - iz_0 s_\alpha \delta_\alpha^\mu \end{matrix}$$

where

$$\mathbf{i}_\beta p_{\alpha\beta} = e^{-\mathbf{s}\mathbf{x}} \partial_\beta \chi_{\alpha\beta}(\mathbf{D}_1) e^{\mathbf{s}\mathbf{x}},\quad \mathbf{i}_\beta p_{\alpha 3} = e^{-\mathbf{s}\mathbf{x}} \partial_{\alpha\beta}^2 \chi_{\alpha\beta}(\mathbf{D}_2) e^{\mathbf{s}\mathbf{x}}$$

$$p_{33} = e^{-\mathbf{s}\mathbf{x}} \partial_{\alpha\beta}^2 \chi_{\alpha\beta}(\mathbf{D}_3) \operatorname{grad} e^{\mathbf{s}\mathbf{x}},\quad p_0 = p_{11} p_{22} - p_{12}^2 \tag{5.2}$$

$$p = p_0 p_{33} - p_{22} p_{13}^2 - p_{11} p_{23}^2 + 2 p_{12} p_{23} p_{13}$$

$$u_\alpha^*(s_1, s_2) = h_\alpha^\mu p^{-1},\quad w^*(s_1, s_2) = h_3^\mu p^{-1} \tag{5.3}$$

$$h_\alpha^\mu = b_{\alpha\beta} T_\beta^* + i b_{\alpha 3} T_3^*,\quad h_3^\mu = b_{33} T_3^* - i b_{\alpha 3} T_\alpha^*$$

$$b_{11} = p_{22} p_{33} - p_{23}^2,\quad b_{12} = p_{13} p_{23} - p_{12} p_{33} \tag{5.4}$$

$$b_{13} = p_{12} p_{23} - p_{22} p_{13},\quad b_{33} = p_0 \quad (1 \leftrightarrow 2)$$

Expressions (5.2) give the homogeneous characteristic polynomials of the corresponding operators in Eqs (2.2); the homogeneous fourth-order polynomials $p_0(s_1, s_2)$, $p_{33}(s_1, s_2)$ are characteristic of the operators of the plane problem and the bending problem with $\mathbf{D}_2 = 0$ and the homogeneous eighth-order polynomial $p(s_1, s_2)$ is characteristic for the entire system (2.2). If $\mu = 1, 2$, $h_\alpha^\mu(s_1, s_2)$ and $h_3^\mu(s_1, s_2)$ are homogeneous polynomials of degree six and five respectively, $h_3^\mu(s_1, s_2)$ and $h_3^3(s_1, s_2)$ are homogeneous polynomials of degree five and four.

Note that, by virtue of the fact that the system of equations (2.2) is elliptic [3], the equation $p(s_1, s_2) = 0$ has no real roots. We will consider the general case of non-repeated characteristic roots λ_k and introduce the notation

$$p(s_1,s_2) = A(\mathbf{D}_1,\mathbf{D}_2,\mathbf{D}_3)\prod_{k=1}^{4}(s_2 - s_1\lambda_k)(s_2 - s_1\bar{\lambda}_k)$$

$$p(1,\lambda_k) = 0,\quad \lambda_k \in C,\quad \operatorname{Im}\lambda_k > 0,\quad A(\mathbf{D}_1,\mathbf{D}_2,\mathbf{D}_3) = \text{const} \in \mathbb{R}$$

$$r_k(s_1,s_2) = -\frac{ip(s_1,s_2)}{s_2 - s_1\lambda_k} = -iA(\mathbf{D}_1,\mathbf{D}_2,\mathbf{D}_3)(s_2 - s_1\lambda_k)\prod_{l\neq k}(s_2 - s_1\lambda_l)(s_2 - s_1\bar{\lambda}_l)$$

$$r_k(1,\lambda_k) = 2(\operatorname{Im}\lambda_k)A(\mathbf{D}_1,\mathbf{D}_2,\mathbf{D}_3)\prod_{l\neq k}(\lambda_k - \lambda_l)(\lambda_k - \bar{\lambda}_l)$$

$$r_k(s_1,s_1\lambda_k) = \frac{\partial p(s_1,s_2)}{\partial s_2}\bigg|_{s_2 = s_1\lambda_k},\quad r_k(1,\bar{\lambda}_k) = -\bar{r}_k(1,\lambda_k)$$

It also follows from Eqs (5.4) that

$$h_\alpha^\mu(1,\bar{\lambda}_k) = \bar{h}_\alpha^\mu(1,\bar{\lambda}_k),\quad h_3^\mu(1,\bar{\lambda}_k) = -\bar{h}_3^\mu(1,\bar{\lambda}_k)\quad (\mu = 1,2)$$

$$h_\alpha^3(1,\bar{\lambda}_k) = -\bar{h}_\alpha^3(1,\bar{\lambda}_k),\quad h_3^3(1,\bar{\lambda}_k) = \bar{h}_3^3(1,\bar{\lambda}_k)$$

We will show that there are originals of (5.1) for displacements u_α^μ, w^μ.

1. We will first assume that $s_1 \in (-\infty, 0]$ and $s_1x'_2 \leqslant 0$ in the inner integrals of (5.1). Integrating with respect to $s_2 \in [-R, R]$, we close the contour of integration in the lower complex half-plane $\operatorname{Im} s_2 \leqslant 0$ by the semi-circle Γ_R: $s_2 = \mathrm{R}e^{i\phi}$. Then, for sufficiently large R (performing the summation from $k = 1$ to $k = 4$, $\zeta_k = x'_1 + \lambda_k x'_2$) we have

$$\left\{\int_{-R}^{R} - \int_{\Gamma_R}\right\}(u_\alpha^\mu)^* e^{-is_2x'_2}ds_2 = -2\pi i\sum_{s_2=\lambda_k s_1}\operatorname{Res}\left\{(u_\alpha^\mu)^* e^{-is_2x'_2}\right\}$$

$$\int_{\Gamma_R}(u_\alpha^\mu)^* e^{-is_2x'_2}ds_2 = O(R^{-1}) \to 0,\quad R \to +\infty\quad (\mu = 1,2)$$

$$\int_{-\infty}^{\infty}\frac{h_\alpha^\mu(s_1,s_2)}{p(s_1,s_2)}e^{-is_2x'_2}ds_2 = -\frac{2\pi}{s_1}\sum\frac{h_\alpha^\mu(1,\lambda_k)}{r_k(1,\lambda_k)}e^{-is_1\lambda_kx'_2} \tag{5.5}$$

$$\int_{-\infty}^{0}\int_{-\infty}^{\infty}(u_\alpha^\mu)^* ds_1ds_2 = -2\pi\sum\frac{h_\alpha^\mu(1,\lambda_k)}{r_k(1,\lambda_k)}\int_{-\infty}^{0}\frac{e^{-is_1\zeta_k}}{s_1}ds_1$$

2. Considering the outer integrals along the ray $s_1 \in (0, +\infty]$, $s_1x'_2 \geqslant 0$ in (5.1), we obtain the conjugate expressions to (5.5). Continuing in this way for the other displacements, we obtain

$$u_\alpha^\mu = -2\sum\operatorname{Re}\left\{c_{\alpha k}^\mu I_1(\zeta_k)\right\},\quad \operatorname{Re}(is_1\zeta_k) \leqslant 0$$

$$\begin{Vmatrix} w^\mu \\ u_\alpha^3 \end{Vmatrix} = -2\sum\operatorname{Re}\left\{i\begin{Vmatrix} c_{3k}^\mu \\ c_{\alpha k}^3 \end{Vmatrix} I_2(\zeta_k)\right\}\quad (\alpha,\mu = 1,2) \tag{5.6}$$

$$w^3 = 2\sum\operatorname{Re}\left\{c_{3k}^3 I_3(\zeta_k)\right\}$$

$$I_n(\zeta_k) = \int_0^\infty\frac{e^{is_1\zeta_k}}{s_1^n}ds_1 = -\frac{(-i\zeta_k)^{n-1}}{(n-1)!}\left\{\ln\zeta_k + \gamma - 1 - \frac{1}{2} - \ \dots\ - \frac{1}{n} + \frac{3\pi i}{2}\right\}$$

$$\begin{Vmatrix} c_{\alpha k}^{\mu} \\ c_{3k}^{\mu} \\ c_{\alpha k}^{3} \\ c_{3k}^{3} \end{Vmatrix} = \frac{1}{2\pi r_k(1,\lambda_k)} \begin{Vmatrix} h_{\alpha}^{\mu}(1,\lambda_k) \\ ih_3^{\mu}(1,\lambda_k) \\ ih_{\alpha}^{3}(1,\lambda_k) \\ -h_3^3(1,\lambda_k) \end{Vmatrix} \tag{5.7}$$

where γ is Euler's constant.

Notice that the contribution to the displacements corresponding to the term $3\pi i/2$ is zero. For this reason, the result is the same for $x'_2 \geqslant 0$ and $x'_2 \leqslant 0$. This can be verified directly and its physical interpretation will be given below. The final expressions for the displacements, angles of rotation of segments, forces and moments will take the form

$$\begin{aligned} &w^3 = 2\sum \operatorname{Re}\{c_{3k}^3 f(\zeta_k)\}, \quad u_{\alpha}^3 = 2\sum \operatorname{Re}\{c_{\alpha k}^3 f'(\zeta_k)\} \\ &\theta_{\alpha}^3 = 2\sum \operatorname{Re}\{\lambda_k^{\alpha-1} c_{3k}^3 f'(\zeta_k)\} \\ &w^{\mu} = 2\sum \operatorname{Re}\{c_{3k}^{\mu} f'(\zeta_k)\}, \quad u_{\alpha}^{\mu} = 2\sum \operatorname{Re}\{c_{\alpha k}^{\mu} f''(\zeta_k)\} \\ &\theta_{\alpha}^{\mu} = 2\sum \operatorname{Re}\{\lambda_k^{\alpha-1} c_{3k}^{\mu} f''(\zeta_k)\} \\ &f(\zeta_k) = \tfrac{1}{2}\zeta_k^2(\ln\zeta_k + \gamma - \tfrac{3}{2}) \end{aligned} \tag{5.8}$$

$$\begin{aligned} &Q_{\alpha\beta}^3 = 2\sum \operatorname{Re}\{q_{\alpha\beta k}^3 f''(\zeta_k)\}, \quad Q_{\alpha 3}^3 = 2\sum \operatorname{Re}\{q_{\alpha 3k}^3 \zeta_k^{-1}\} \\ &Q_{\alpha\beta}^{\mu} = 2\sum \operatorname{Re}\{q_{\alpha\beta k}^{\mu} \zeta_k^{-1}\}, \quad Q_{\alpha 3}^{\mu} = 2\sum \operatorname{Re}\{q_{\alpha 3k}^{\mu} \zeta_k^{-2}\} \\ &\left(Q_{\alpha\beta}^3, Q_{\alpha\beta}^{\mu} \leftrightarrow M_{\alpha\beta}^3, M_{\alpha\beta}^{\mu};\ q_{\alpha\beta k}^3, q_{\alpha\beta k}^{\mu} \leftrightarrow m_{\alpha\beta k}^3, m_{\alpha\beta k}^{\mu}\right) \end{aligned} \tag{5.9}$$

where $q_{\alpha\beta k}$, $m_{\alpha\beta k}$, $q_{\alpha 3k}$ are rational-fractional functions of the characteristic roots λ_k obtained by substituting the displacements (5.8) into expressions (2.4). At the point $\mathbf{x} = \mathbf{x}_0$ the displacements $u^{\mu}_{\alpha}(\mathbf{x}, \mathbf{x}_0)$, angles of rotation $\theta^{\mu}_{\alpha}(\mathbf{x}, \mathbf{x}_0)$, forces and moments $Q^3_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0)$ and $M^3_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0)$ have an integrable (logarithmic) singularity. The functions $u^{\mu}_n = n_{\alpha}u^{\mu}_{\alpha}$, $u^{\mu}_{\tau} = \tau_{\alpha}u^{\mu}_{\tau}$, $Q^3_n = n_1^2 Q^3_{11} + 2n_1n_2Q^3_{12} + n_2^2Q^3_{12} = n_2^2Q^3_{22}$, $Q^3_{\tau} = n_1n_2(Q^3_{22} - Q^3_{11}) + (n_1^2 - n_2^2)Q^3_{12}$ ($u \leftrightarrow \theta$, $Q \leftrightarrow M$, $\mathbf{n} = \mathbf{i}_{\alpha}n_{\alpha}$, $\boldsymbol{\tau} = \mathbf{i}_{\alpha}\tau_{\alpha}$) behave in exactly the same way. The functions $Q^{\mu}_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0)$, $M^{\mu}_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0)$ and $Q^3_{\alpha 3}(\mathbf{x}, \mathbf{x}_0)$ (like the functions $Q^3_{n3} = n_{\alpha}Q^3_{\alpha 3}$, $P^3_n(\mathbf{x}, \mathbf{x}_0)$) have a simple pole; the functions $Q^{\mu}_{\alpha 3}(\mathbf{x}, \mathbf{x}_0)$ and $P^{\mu}_n(\mathbf{x}, \mathbf{x}_0)$ have a second-order pole.

The second-order singularities appear because it is an associated bending and tension–compression–shear problem. They do not arise for a symmetrically-stratified plate $\mathbf{D}_2 = 0$, $z_0 = 0$. This creates an additional difficulty and requires special investigation.

6. Generally speaking, the fundamental solutions (5.8) are determined to the accuracy of any solution of Eqs (2.2) with zero right-hand side. Solutions of a homogeneous system of equations of this kind can be obtained by the method of complex potentials [3, 5], for instance. The fact that formulae (5.8) contain multivalued (logarithmic) functions is important. The coefficients $c^{\mu}_{\alpha k}$, c^{μ}_{3k}, $c^3_{\alpha k}$, c^3_{3k} in the logarithms are also completely determined by the method of potentials starting from the condition that the displacements, angles of rotation, deformations and curvatures (or longitudinal forces and moments) are single-valued, provided that the principal vector and principal moment of individual loads (4.1) and boundary loads (for the solution (5.8)) correspond on any closed contour that includes the point $\mathbf{x}_0$. The expressions for the constants are the same as (5.7). Thus, the kernels of Eqs (4.2) are single-valued functions, and the logarithmic components in the solutions (5.8) give the simplest and minimal set of multivalued functions needed to construct them.

A detailed general analysis of the 20 relations of one-to-one correspondence is given in [5]. We give some of these for loads (4.1)

$$4\pi\sum \operatorname{Re}\left\{ i \begin{Vmatrix} q_{12k}^{\mu} \\ q_{22k}^{\mu} \end{Vmatrix} \right\} = \begin{Vmatrix} \delta_1^{\mu} \\ \delta_2^{\mu} \end{Vmatrix}, \quad 4\pi\sum \operatorname{Re}\{iq_{23k}^{\mu}\} = \delta_3^{\mu}$$

$$4\pi\sum \mathrm{Re}\left\{i\begin{Vmatrix}\lambda_k^{-1}m_{11k}^{\mu}\\ m_{22k}^{\mu}\end{Vmatrix}\zeta_k^{\delta_\mu^3}\right\}=z_0\begin{Vmatrix}-\delta_1^{\mu}\\ \delta_2^{\mu}\end{Vmatrix}\quad(\mu=1,2,3) \tag{6.1}$$

$$\sum \mathrm{Re}\left\{i\zeta_k^2 c_{3k}^3\right\}=0,\quad \sum \mathrm{Re}\left\{i\zeta_k c_{\alpha k}^{\mu}\right\}=0,\quad \sum \mathrm{Re}\left\{ic_{\alpha k}^{\mu}\right\}=0$$

The last three equations have already been taken into account in formulae (5.8) in the transforms $I_n(\zeta_k)$ of the generalized functions; they express the fact that the displacements and angles of rotation are single-valued.

7. To transform Eqs (4.2) with kernels (5.8), we will prove an auxiliary assertion. We consider a singular integral over the sufficiently smooth contour $\partial\Omega$ of a simply-connected region Ω (l is an arbitrary arc coordinate)

$$F_{mk}(f,\mathbf{x}_0)=\int_{\partial\Omega}\frac{f(\mathbf{x})}{\zeta_k^m}dl,\quad \mathbf{x}\in\partial\Omega,\quad \mathbf{x}_0\notin\partial\Omega$$

where $f(\mathbf{x})$, $\partial_l^{m_1}f(m_1=1,2,\ldots,m-1)$ are real functions of the Hölder class on $\partial\Omega$ which have a smooth continuation in the region $\Omega_\pm$ remaining on the left (right) when $\partial\Omega$ is traversed in a positive direction. We introduce the functions $g_k(\mathbf{x})=(\tau_1+\lambda_k\tau_2)^{-1}$, $\boldsymbol{\tau}=\mathbf{i}_\alpha\tau_\alpha$, the unit tangent vector at the point $\mathbf{x}\in\partial\Omega$.

Lemma. The limiting behaviour of the function

$$F_{mk}^{\mp}(f,\mathbf{x}_0)=F_{mk}(f,\mathbf{x}_0),\quad \mathbf{x}_0\in\Omega_\mp,\quad \mathbf{x}_0\to\mathbf{x}_0\in\partial\Omega$$

is governed by the equations (V.p. is the principal value of the integral)

$$(m-1)F_{mk}^{\mp}(f,\mathbf{x}_0)=F_{m-1k}^{\mp}(\partial_\tau F_k(\mathbf{x}),\mathbf{x}_0) \tag{7.1}$$

$$F_{1k}^{\mp}(f,\mathbf{x}_0)=\pi i\left\{\mp F_k(\mathbf{x}_0)+\mathrm{V.p.}\int_{\partial\Omega}\frac{f(\mathbf{x})}{\zeta_k}dl\right\}$$

$$F_{2k}^{\mp}(f,\mathbf{x}_0)=\pi i\left\{\mp\partial_\tau(F_k(\mathbf{x}_0))g_k(\mathbf{x}_0)+\mathrm{V.p.}\int_{\partial\Omega}\frac{\partial_\tau F_k(\mathbf{x})}{\zeta_k}dl\right\}$$

$$F_k(\mathbf{x})\equiv f(\mathbf{x})g_k(\mathbf{x})$$

The proof follows from the Sokhotskii–Plemelj [7] theorem and the equations

$$\frac{dl}{d\zeta_k}=g_k(\mathbf{x}),\quad \frac{f(\mathbf{x})}{\zeta_k^m}=\frac{1}{m-1}\left\{\zeta_k^{1-m}\partial_\tau F_k(\mathbf{x})-\partial_\tau\left(F_k(\mathbf{x})\zeta_k^{1-m}\right)\right\}$$

8. Suppose, to fix our ideas, that the region $\Omega=\Omega_+$ in Eqs (4.2) is finite and simply-connected and has a sufficiently smooth boundary. Let the point $\mathbf{x}_0\in\Omega_+$ approach the boundary of the region. Using representations (5.8) and (5.9), formulae (7.1), relations (6.1) and the equations ($\mu=1,2,3$)

$$q_{11k}^{\mu}=-\lambda_k q_{12k}^{\mu}=\lambda_k^2 q_{22k}^{\mu},\quad q_{13k}^{\mu}=-\lambda_k q_{23k}^{\mu},\quad q_{\alpha 3k}^{\mu}=m_{\alpha 1k}^{\mu}+\lambda_k m_{\alpha 2k}^{\mu}$$

finally from Eqs (4.2) we obtain

$$w(\mathbf{x}_0)=\int_{\Omega}\left\{u_\alpha^3 T_\alpha+w^3T_3\right\}d\Omega+\sum\mathrm{Re}\left\{2\pi i\int_{\partial\Omega}\frac{w(\mathbf{x})q_{23k}^3}{\zeta_k}d\zeta_k\right\}+$$

$$+\int_{\partial\Omega}\left\{u_n^3Q_n+u_\tau^3Q_\tau+\theta_n^3M_n+w^3P_n-u_nQ_n^3-u_\tau Q_\tau^3-\theta_nM_n^3-wP_n^3\right\}dl \tag{8.1}$$

$$\theta_\alpha(\mathbf{x}_0) = -\int_\Omega \partial^\circ_\alpha\left\{u^3_\beta T_\beta + w^3 T_3\right\}d\Omega - \int_{\partial\Omega}\partial^\circ_\alpha\left\{u^3_n Q_n + u^3_\tau Q_\tau + \theta^3_n M_n + w^3 P_n\right\}dl +$$

$$+\sum \mathrm{Re}\left\{2\pi i\lambda_k^{\alpha-1}\int_{\partial\Omega}\left(\frac{q^3_{11k}}{\lambda_k}\upsilon_1 - q^3_{22k}\upsilon_2 + \frac{t^3_{11k}}{\lambda_k}\theta_1 - t^3_{22k}\theta_2\right)\frac{d\zeta_k}{\zeta_k}\right\} \tag{8.2}$$

$$\upsilon_\alpha(\mathbf{x}_0) = \int_\Omega\left\{u^\alpha_\beta T_\beta + w^\alpha T_3\right\}d\Omega + \int_{\partial\Omega}\left\{u^\alpha_n Q_n + u^\alpha_\tau Q_\tau + \theta^\alpha_n M_n + w^\alpha P_n\right\}dl +$$

$$+\sum \mathrm{Re}\left\{2\pi i\int_{\partial\Omega}\left(-\frac{q^\alpha_{11k}}{\lambda_k}\upsilon_1 + q^\alpha_{22k}\upsilon_2 - \frac{t^\alpha_{11k}}{\lambda_k}\theta_1 + t^\alpha_{22k}\theta_2\right)\frac{d\zeta_k}{\zeta_k}\right\} \tag{8.3}$$

$$\upsilon_\alpha \equiv u_\alpha + z_0\theta_\alpha,\quad t^\mu_{\alpha\beta k} \equiv m^\mu_{\alpha\beta k} - z_0 q^\mu_{\alpha\beta k}\quad \left(\partial^\circ_\alpha \equiv \frac{\partial}{\partial x^\circ_\alpha}\right) \tag{8.4}$$

In the regularized form in Eqs (8.1)–(8.3), Cauchy-type singular integrals can be understood in the sense of the principal value; the left-hand sides of Eqs (8.1) and (8.3) must then be replaced by $1/2w(\mathbf{x}_0)$ and $1/2\upsilon_\alpha(\mathbf{x}_0)$, respectively; the left-hand side of Eq. (8.2) will not change.

The first boundary-value problem reduces to a system of Fredholm integral equations of the first kind (8.2), (8.3) in the unknown forces and moments. The kernels of the equations have a weak singularity; all the integrals are convergent.

The second boundary-value problem reduces to a system of singular integral equations (8.2), (8.3) in the unknown displacements υ_α of the mid-plane $z = z_0$ in plan, and angles of rotation θ_α of the sections. In the optimum system of coordinates, where the norm of the operator of the association of the processes of bending and tension–compression–shear is a minimum, generally speaking $z_0 \neq 0$ [3].

It is important to emphasize that although (according to (5.9)) the kernels of Eqs (4.2) have a higher-order singularity than in the classical plane (bending) problem, the extra derivatives of the displacements of type (7.2) in Eqs (8.3) disappear because of the conditions for one-to-one correspondence between the principal vector and the principal load moment.

9. We will show that the resulting system of BIE has a zero index, or is quasi-Fredholm [7, 8]. We will denote the arc coordinate of the point $\mathbf{x}_0 \in \partial\Omega$ by $l_0 = l_0(\mathbf{x}_0)$, $\zeta = x'_1 + ix'_2$, $\phi = \arg\zeta$ and introduce the functions

$$f_k(\mathbf{x}) = (\lambda_k - i)\bar{\zeta}\zeta_k^{-1},\quad g(\mathbf{x}) = \frac{d\zeta}{dl}\,\frac{l-l_0}{\zeta} - 1$$

$$\frac{d\zeta_k}{\zeta_k} = \frac{d\zeta}{\zeta} + f_k(\mathbf{x})d\phi = \frac{dl}{l-l_0} + g(\mathbf{x})dl + f_k(\mathbf{x})d\phi \tag{9.1}$$

which are continuous on $\partial\Omega$. We then reduce the system of equations (8.2) and (8.3) to the following form

$$\mathbf{E}\mathbf{y}(\mathbf{x}_0) + \mathbf{A}\int_{\partial\Omega}\frac{\mathbf{y}(\mathbf{x})}{l-l_0}dl + \mathbf{H}_1(\mathbf{y}) = \mathbf{F}(\mathbf{x}_0) \tag{9.2}$$

$$\mathbf{E}\mathbf{y}(\mathbf{x}_0) + \mathbf{A}\int_{\partial\Omega}\frac{\mathbf{y}(\mathbf{x})}{\zeta}d\zeta + \mathbf{H}_2(\mathbf{y}) = \mathbf{F}(\mathbf{x}_0) \tag{9.3}$$

where

$$\mathbf{y} = (\theta_1, \theta_2, \upsilon_1, \upsilon_2)^T$$

$$\mathbf{A}_k = \begin{Vmatrix} -\lambda_k^{-1}q^3_{11k} & q^3_{22k} & -\lambda_k^{-1}t^3_{11k} & t^3_{22k} \\ -q^3_{11k} & \lambda_k q^3_{22k} & -t^3_{11k} & \lambda_k t^3_{22k} \\ \lambda_k^{-1}q^1_{11k} & -q^1_{22k} & \lambda_k^{-1}t^1_{11k} & -t^1_{22k} \\ \lambda_k^{-1}q^2_{11k} & -q^2_{22k} & \lambda_k^{-1}t^2_{11k} & -t^2_{22k} \end{Vmatrix}$$

$$\mathbf{A} = \operatorname{diag}(0, 0, -\tfrac{1}{2}, -\tfrac{1}{2}) = -\sum \operatorname{Re}(2\pi i \mathbf{A}_k)$$

$\mathbf{E}$ is the identity matrix, the operators $\mathbf{H}_1(\mathbf{y})$, $\mathbf{H}_2(\mathbf{y})$ are completely continuous in the space $L_2(\partial\Omega)$ (by virtue of the fact that the kernels of (9.1) are continuous), and $\mathbf{F}(\mathbf{x}_0)$ is the right-hand side of the equations, defined by the given boundary forces and moments on $\partial\Omega$.

Then for the index of the system of singular equations [7] (the increment of the argument of the complex function $\det(\mathbf{E} - \mathbf{A})/\det(\mathbf{E} + \mathbf{A})$ when $\partial\Omega$ is traversed in a positive direction normalized by 2π) we obtain

$$\operatorname{ind} \equiv \frac{1}{2\pi}\left[\arg\frac{\det(\mathbf{E}-\mathbf{A})}{\det(\mathbf{E}+\mathbf{A})}\right]_{\partial\Omega} = 0, \quad \det(\mathbf{E} \mp \mathbf{A}) \neq 0$$

The four BIE (8.2), (8.3) or (9.2), (9.3) belong to the simplest class of systems of singular equations with a zero index, all of whose Fredholm alternatives are satisfied [7, 8].

10. The equations for a plate which occupies the exterior region $\Omega = \Omega_-$ in plan are analysed in exactly the same way. The only difference in Eqs (8.2), (8.3) or (9.2), (9.3) is the direction in which the contour is traversed and the signs of the corresponding integrals.

11. We will now point out some properties of the fundamental solutions (5.8). From the form of the Fourier transforms (5.3) and (5.4) it can be shown that

$$\mathbf{M}(\mathbf{x}, \mathbf{x}_0) \equiv \left\| \begin{matrix} v_1^1 & v_2^1 & w^1 \\ v_1^2 & v_2^2 & w^2 \\ -v_1^3 & -v_2^3 & -w^3 \end{matrix} \right\|, \quad \mathbf{M}^T = \mathbf{M} \tag{11.1}$$

Note that we have omitted the coefficient ε^{-1} in front of the longitudinal loads (2.1), (4.1) for brevity; generally speaking $w^\alpha = -\varepsilon^{-1} v_\alpha^3$. It is not particularly important for the loads in (4.1) assigned on two front surfaces to be symmetric. For example, if we put

$$\tau_\alpha^+ = \delta_\alpha^\mu \delta(\mathbf{x}'), \quad \sigma^+ = \delta_3^\mu \delta(\mathbf{x}'), \quad \sigma^- = 0, \quad \tau^- = 0$$

$$T_\alpha = \delta_\alpha^\mu \delta'(\mathbf{x}'), \quad T_3 = \delta_3^\mu \delta(\mathbf{x}') + z^+ \delta_\alpha^\mu \partial_\alpha \delta(\mathbf{x}')$$

we need only replace z_0 by z^+ in the total BIE, and the longitudinal displacements v_α^μ will correspond to the front surface $z = z^+$.

Equations (11.1) define the symmetric Green's tensor in the statics of multilayer plates with arbitrarily arranged anisotropic layers. If the layers are symmetrically stacked $z_0 = 0$, the mixed stiffnesses $\mathbf{D}_2 = 0$, the displacements $w^\alpha = v_\alpha^3 = 0$ and $p = p_0 p_{33}$, that is, the eigenvalues λ_k split into independent pairs for the bending and plane problems. In that case the fundamental solutions are the same as those obtained in [8, 13].

12. We conclude by listing our main results. We have constructed a state of fundamental solutions of the problem of the statics of an asymmetrically multilayered anisotropic plate, distinguished by the fact that the processes of bending and tension–compression–shear are associated. On the basis of the reciprocity theorem, four BIE (8.2), (8.3) of the basic boundary-value problems have been identified. The components of the contour integrals on the right-hand sides of the BIE can, by analogy with the classical problems [8, 11], be called the potentials of the displacements, angle of rotation, longitudinal forces, and bending and shear potentials of the plate, respectively. The first boundary-value problem has been reduced to Fredholm equations of the second kind. The integral equations of the remaining boundary-value problems are obtained from corresponding combinations of BIE (8.1)–(8.3). Although formally the fundamental solutions yield second-order singularities, the form of the resulting equations is the same as in the well-known BIE for the classical bending of single plates [11, 13] and the plane problem [8, 10, 11]. The main difference is that the dimension is increased, since associated processes are being considered.

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